CSci 5304, F01	Final exam	Dec. 18th, 2001

- Open book/notes exam. Calculators OK.
- Duration: 90mn
- There are three questions to answer. In all cases, brief justifications of your answers are required (no long proofs).
- The weight of each question is indicated in the brackets **before** the question. Base is: 100pts.

---

1. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 2 & -i & 0 \\ i & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Give brief answer to the following questions (no proofs)

(a) [**4 pts**] Is A symmetric positive definite?

(b) [**4 pts**] Will the standard power method converge for A (Given an arbitrary nonzero initial vector)?

(c) [**4 pts**] Will the standard power method converge for B (Given an arbitrary nonzero initial vector)?

(d) [**4 pts**] Use Gershgorin's theorem to find a region where all the eigevalues of C lie.

(e) [**4 pts**] Will the power method converge for C (Given an arbitrary nonzero complex initial vector)?

(f) [**4 pts**] Find all (real) scalars α such that the sequence of matrices $B_k = (\alpha B)^k$ converges to zero when $k \to \infty$.

2. Consider the linear system $Ax = b$ where

$$A = \begin{pmatrix} 12 & -6 & 4 \\ 6 & -4 & 3 \\ 4 & -3 & 2.4 \end{pmatrix} \quad b = \begin{pmatrix} 10 \\ 5 \\ 3.5 \end{pmatrix}$$

The condition number of A associated with the 1-norm is $\kappa_1(A) = 748$.

(a) Consider the vector $\tilde{x} = (1, 1, 1)^T$ as an approximate solution to the system. [**4 pts**] Calculate the residual norm $\|b - A\tilde{x}\|_1$. [**6 pts**] Use the condition number and this residual norm to give an upper bound for the norm $\|x - \tilde{x}\|_1 / \|x\|_1$, where x is the exact solution of the system.

(b) [**7 pts**] Show that if the term $a_{33} = 2.4$ is replaced by $a_{33} = 2.333... = 7/3$ the matrix A becomes singular. [Hint: multiply the new A by $[1, 6, 6]^T$]

(c) [**8 pts**] Use the result of the previous question to find a lower bound for $\kappa_1(A)$. Compare with the condition number given above and verify that you do indeed obtain a lower bound.

3. Consider the matrix

$$A = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ \gamma & 0 & 1 \end{pmatrix}$$

in which the scalars α, β, γ are all real positive.

(a) [**5 pts**] Take $\alpha = \beta = \gamma = \frac{1}{2}$. Use Gershgorin's theorem to find a region in the complex plane where all eigenvalues of A are located.

(b) [**8 pts**] Find all the eigenvalues of A (α, β, γ are no longer equal to $1/2$).

(c) [**8 pts**] Assume that the Jacobi iteration is used to solve a system with A. Under what conditions on α, β, γ will the iteration converge?

(d) [**8 pts**] Assume that the Gauss-Seidel iteration is used to solve a system with A. Under what conditions on α, β, γ will the iteration converge?

(e) [**7 pts**] In the case where $\alpha = \beta = \gamma = 0.5$ how would you compare the rates of convergence of the Jacobi and Gauss-Seidel iterations?

4. Let S be an $n \times m$ matrix of full rank and consider

$$A = \begin{pmatrix} \alpha I & S \\ S^T & -\beta I \end{pmatrix}$$

(a) [**7 pts**] Assuming $\alpha > 0$, $\beta > 0$, what is the inertia of the matrix A?

(b) Assuming now that $\alpha > 0$, $\beta < 0$, and $-\alpha\beta = \sigma_k^2$ where σ_k is the k-th singular value of S [**8 pts**] What is the inertia of A?